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A powerful approach to generate new integrable systems

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Abstract. In this paper, an effective algorithm to generate integrable systems is given. As a result, many new integrable equations are derived in a systematic way.

1. Introduction

It is known that a central and very active topic in the theory of integrable systems is the search for as many new integrable systems as possible. A key feature of an integrable equation is the fact that it can be expressed as the compatibility condition of two suitable linear equations, usually referred to as a Lax pair. One of these equations is time-independent, takes the form of a linear eigenvalue problem for an eigenfunction ψ and eigenvalue λ, and plays a crucial role in the inverse scattering transform. On the other hand, the role of Lie algebras in the integrable systems has attracted much attention [1–6]. Although the various ways in which Lie algebras have entered into the theory are not entirely identical, Lie algebras essentially have served as the ground in which the principal elements of Lax and zero-curvature equations grow—at least this is so in two theories which have been systematically developed: Wilson's general zero-curvature associated with simple Lie algebras [1], and the theory of Drinfel'd and Sokolov of equations associated with affine Kac–Moody Lie algebras [2].

Recently motivated by Wilson and Drinfel'd and Sokolov's idea, Tu considered a model isospectral problem and proposed, by introducing modified quantities, a new method to generate integrable systems [7]. Furthermore, Tu proposed a new approach to Hamiltonian structures of integrable systems—Trace identity [8].

In this paper, we develop Tu's approach and give a very effective algorithm to generate integrable systems. As a result, many known and unknown integrable systems are derived in such a unified way.

This paper is divided into four sections. In the next section, we first introduce some notation and conventions. A very effective algorithm to generate integrable systems is given, which results in many new integrable systems being derived in a systematic and unified way. Some illustrative examples are also given. In section 3, some other interesting examples are considered. Finally, some concluding remarks are given in section 4.

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2. An algorithm to generate integrable systems

Let us begin with the following model isospectral problem:

$$\psi_x = U\psi \tag{1}$$

where $U = e_0(\lambda) + u_1 e_1(\lambda) + \ldots + u_p e_p(\lambda)$, $e_i(\lambda)$ $(i=0, 1, \ldots, p)$ belongs to a loop algebra $\mathscr{G} = sl(N, C) \otimes C[\lambda, \lambda^{-1}]$, and u_i $(i=1, \ldots, p)$ is taken from the Swartz space $\mathscr{S}(-\infty, \infty)$. Different gradations of $\mathscr{G}$ may be available;

$$\deg(X \otimes \lambda^n) = n \qquad \text{for } X \in sl(N, C). \tag{2}$$

Throughout the paper, we always fix the gradation (2). Just as in [7–10], we assume that $e_i(\lambda)$ $(i=0, \ldots, p)$ meets the conditions:

(i) $e_0(\lambda), e_1(\lambda), \ldots, e_p(\lambda)$ are linearly independent;

(ii) $e_0(\lambda)$ is pseudoregular, i.e.

$$\mathscr{G} = \mathrm{Kerad}\, e_0(\lambda) \oplus \mathrm{Imad}\, e_0(\lambda)$$

$$\mathrm{Kerad}\, e_0(\lambda) \text{ is commutative}$$

where

$$\mathrm{Kerad}\, e_0(\lambda) = \{X \mid X \in \mathscr{G}, [X, e_0(\lambda)] = 0\}$$

$$\mathrm{Imad}\, e_0(\lambda) = \{Y \in \mathscr{G}, \text{ s.t. } X = [Y, e_0(\lambda)] = 0\}$$

(iii) $d_0 > 0$, $d_0 > d_1 \geqslant d_2 \geqslant \ldots \geqslant d_p$

where $d_i = \deg e_i(\lambda)$.

Note that $e_0(\lambda) = e_0 \otimes \lambda^{d_0}$ with $e_0 \in sl(N, c)$. It is easy to deduce that

$$sl(N, C) = \mathrm{Kerad}\, e_0 \oplus \mathrm{Imad}\, e_0$$

$$\mathrm{Kerad}\, e_0 \text{ is commutative.}$$

Further, we can easily show the following proposition.

Proposition 1. Matrix e_0 is similar to a diagonal matrix with N distinct diagonal elements.

Due to this result, we set e_0 as a diagonal matrix with N distinct diagonal elements without loss of generality in the following discussion, i.e.

$$e_0 = \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_N \end{bmatrix} \qquad \alpha_i \neq \alpha_j \, (i \neq j) \sum_{i=1}^{N} \alpha_i = 0.$$

Before going into details, we first recall Tu's scheme for generating integrable systems. The scheme contains two steps. First we take a solution V of the co-adjoint equation associated with (1),

$$V_x = [U, V] \tag{3}$$

which plays a key role in generating integrable systems and Hamiltonian structures [7–10]. Secondly, we search for a $\Delta_n \in \mathscr{G}$ such that for

$$V^{(n)} = (\lambda^n V)_+ + \Delta_n$$

the following holds:

$$-V_x^{(n)} + [U, V^{(n)}] \in Ce_1(\lambda) + Ce_2(\lambda) + \ldots + Ce_p(\lambda).$$

This requirement yields a hierarchy of evolution equations:

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0.$$

Obviously, Tu's scheme for generating integrable systems depends on the existence of V in (3) and choice of $V^{(n)}$. In [9], Tu proved that (3) has a series solution $V = \Sigma_{k=0}^{\infty} V^{(k)}$, deg $V^{(k)} = -k$ under the extra assumption

$$d_0 > 2d_1 \tag{4}$$

where $U, V \in \tilde{G} = G \otimes C[\lambda, \lambda^{-1}]$, G is a finite-dimensional Lie algebra over C and the gradation for $\tilde{G}$ is not limited to natural gradation. In our present case: $G = sl(N, C)$, and as the gradation is defined by (2), we can remove the assumption (4) and get the following result.

Proposition 2. There exists a non-zero $V = \Sigma_{i=0}^{\infty} V_i \lambda^{-i} \in \mathscr{G}$ such that (3) holds and elements of matrix V_i are all pure polynomials of u_i $(i = 1, \ldots, p)$ and their derivatives.

Proof. The proof of proposition 2 contains two steps. First we prove that there exists a non-zero $V = \Sigma_{i=0}^{\infty} V_i \lambda^{-i}$ such that

$$V_{\mathrm{F}_x} = [U, V]_{\mathrm{F}} \tag{3'}$$

$$(\mathrm{tr}\ V^k)_x = 0 \qquad k = 1, \ldots, N$$

and elements of matrix V_i are all pure polynomials of u_i $(i = 1, \ldots, p)$ and their derivatives, where $V_{\mathrm{F}} = V - V_{\mathrm{D}}$ with V_{D} representing the diagonal part of V and tr is a trace of matrix. The proof of this part is standard. We substitute

$$V = \sum_{i=0}^{\infty} V_i \lambda^{-i} \equiv \sum_{i=0}^{\infty} \sum_{j,k=1}^{N} V_{jk}^{(i)} E_{jk} \lambda^{-i}$$

with the initial values $V_{jk}^{(0)} = \ldots = V_{jk}^{(d_0 - d_1 - 1)} = 0$ $(j \neq k)$, $V_{ii}^{(0)} = \beta \alpha_i$ $(i = 1, \ldots, N)$ into (3′); here $E_{jk} \equiv (\delta_{ij}\delta_{lk})_{1 \leqslant i,l \leqslant N}$, β is a non-zero constant and

$$U = \sum_{l=1}^{N} \alpha_l E_{ll} \lambda^{d_0} + \sum_{k=1}^{p} \sum_{i,j=1}^{N} \alpha_{ij}^{(k)} E_{ij} u_k \lambda^{d_k}.$$

We have

$$(\alpha_i - \alpha_j) V_{ij}^{(k+d_0)} = V_{ij}^{(k)}{}_x + \sum_{l=1}^{N} \sum_{k'=1}^{p} (V_{il}^{(k+d_{k'})} \alpha_{lj}^{(k')} - V_{lj}^{(k+d_{k'})} \alpha_{il}^{(k')}) \qquad i \neq j \qquad k \geqslant -d_1 \tag{5a}$$

$$\sum_{\substack{1\leqslant i_1,\ldots,i_l\leqslant N\\ n_1+\ldots+n_l=n}} V_{i_1i_2}^{(n_1)}\ldots V_{i_{l-1}i_l}^{(n_{l-1})}V_{i_li_1}^{(n_l)}=\text{const} \tag{5b}$$

$$n=1,2,\ldots \qquad l=1,2,\ldots,N.$$

Here we have followed the following convention for the sake of convenience:

$$V_{ij}^{(m)}\equiv 0 \qquad m<0 \qquad 1\leqslant i,j\leqslant N.$$

It is easy to show that all the $V_{ij}^{(k)}$ can be determined by the recursion relations step by step and $V_{ij}^{(k)}$ are all pure polynomials of u_i $(i=1,\ldots,p)$ and their derivatives by induction. In fact, suppose that $V_{ij}^{(k)}$ $(i\neq j)$, $0\leqslant k\leqslant m-d_1-1$ and $V_{ii}^{(k)}$, $0\leqslant k\leqslant m-d_0$ are the known pure polynomials of u_i $(i=1,\ldots,p)$ and their derivatives, then by taking $k=m-d_0-d_1$ in (5*a*), we can easily deduce from (5*a*) that $V_{ij}^{(m-d_1)}$ $(i\neq j)$ is also a known pure polynomial of u_i $(i=1,\ldots,p)$ and their derivatives. Further, a detailed analysis of (5*b*) with $n=m-d_0+1$ implies that

$$\sum_{i=1}^{N}(\beta\alpha_i)^{l-1}V_{ii}^{(m-d_0+1)}=F_l \qquad l=1,\ldots,N \tag{6}$$

where F_l $(l=1,\ldots,N)$ is some known pure polynomial of $V_{ij}^{(k)}$ $(i\neq j$, $1\leqslant i,j\leqslant N, 0\leqslant k\leqslant m-d_1-1)$ and $V_{ii}^{(k)}$ $(i=1,\ldots,N, 0\leqslant k\leqslant m-d_0)$. From (6), we immediately deduce that $V_{ii}^{(m-d_0+1)}$ $(i=1,\ldots,N)$ can be uniquely determined and is a pure polynomial of u_j $(j=1,\ldots,p)$ and their derivatives. The second step of the proof is to show that V so obtained satisfies (3). Set

$$B\equiv V_x-[U,V].$$

It suffices to show that $B=0$. From (3′), we know that B must be a diagonal matrix and

$$\operatorname{tr}(V^kB)=0 \qquad k=0,1,\ldots,N-1. \tag{7}$$

In fact,

$$\begin{aligned}\operatorname{tr}(V^kB)&=\operatorname{tr}(V^k(V_x-[U,V]))\\ &=\operatorname{tr}(V^kV_x)-\operatorname{tr}([U,V^{k+1}])\\ &=\operatorname{tr}(V^kV_x)=\frac{1}{k+1}(\operatorname{tr}V^{k+1})_x=0.\end{aligned}$$

Substitution of

$$B=\sum_{k=-d_0}^{\infty}\sum_{i=1}^{N}B_{ii}^{(k)}E_{ii}\lambda^{-k}$$

into (7) leads us to

$$\sum_{-d_0\leqslant k\leqslant n}\sum_{i_1=1}^{N}\sum_{\substack{1\leqslant i_1,\ldots,i_l\leqslant N\\ j_1+\ldots+j_l=n-k}} V_{i_1i_2}^{(j_1)}\ldots V_{i_{l-1}i_l}^{(j_{l-1})}V_{i_li_1}^{(j_l)}B_{i_1i_1}^{(k)}=0 \tag{8a}$$

$$n=-d_0,-d_0+1,\ldots \qquad 1\leqslant l\leqslant N-1$$

$$\sum_{i=1}^{N}B_{ii}^{(k)}=0 \qquad k\geqslant -d_0. \tag{8b}$$

When $n=-d_0$, (8) becomes

$$\sum_{i=1}^{N}(\beta\alpha_i)^l B_{ii}^{(-d_0)}=0 \qquad l=0,1,\ldots,N-1$$

which implies that

$$B_{ii}^{-d_0}=0 \qquad i=1,\ldots,N.$$

Now suppose that $B_{ii}^{(k)}=0$ $(-d_0\leqslant k\leqslant m)$, then (8) with $n=m+1$ leads to

$$\begin{aligned}0&=\sum_{-d_0\leqslant k\leqslant m+1}\sum_{i_1=1}^{N}\sum_{\substack{1\leqslant i_1,\ldots,i_l\leqslant N\\ j_1+\ldots+j_l=m+1-k}} V_{i_1i_2}^{(j_1)}\ldots V_{i_{l-1}i_l}^{(j_{l-1})}V_{i_li_1}^{(j_l)}B_{i_1i_1}^{(k)}\\ &=\sum_{-d_0\leqslant k\leqslant m}\sum_{i_1=1}^{N}\sum_{\substack{1\leqslant i_1,\ldots,i_l\leqslant N\\ j_1+\ldots+j_l=m+1-k}} V_{i_1i_2}^{(j_1)}\ldots V_{i_{l-1}i_l}^{(j_{l-1})}V_{i_li_1}^{(j_l)}B_{i_1i_1}^{(k)}+\sum_{i_1=1}^{N}(V_{i_1i_1}^{(0)})^l B_{i_1i_1}^{(m+1)}\\ &=\sum_{i=1}^{N}(\beta\alpha_i)^l B_{i_1i_1}^{(m+1)} \qquad l=1,\ldots,N-1\end{aligned}$$

and

$$\sum_{i=1}^{N}B_{ii}^{(m+1)}=0$$

which imply that $B_{ii}^{(m+1)}=0$ $(i=1,\ldots,N)$. Thus, we have proved by induction that all the $B_{ii}^{(k)}$ vanish, i.e. $B=0$. Therefore the proof of proposition 2 is completed.

This result is important as assumption (4) is restricted. For example, consider the Kaup–Newell spectral problem

$$\psi_x=\begin{pmatrix}-\mathrm{i}\lambda^2 & \lambda q\\ \lambda r & \mathrm{i}\lambda^2\end{pmatrix}\psi.$$

In this case, $d_0=2$, $d_1=d_2=1$, and then (4) is not satisfied. More importantly, due to the removal of the extra assumption (4) in proposition 2, we can determine that, for which kind of U we are able to choose $V^{(n)}=(\lambda^nV)_+$. Thus, an effective algorithm to generate integrable systems is given. In fact, from proposition 2, we deduce that

$$-(\lambda^nV)_{+x}+[U,(\lambda^nV)_+]=(\lambda^nV)_{-x}-[U,(\lambda^nV)_-] \tag{9}$$

where $(\lambda^nV)_+\equiv\Sigma_{i=0}^{n}V_i\lambda^{n-i}$ and $(\lambda^nV)_-=\lambda^nV-(\lambda^nV)_+$. It is easy to see that the terms on the left side of (9) are of degrees not less than $d_p^-\equiv(d_p-|d_p|)/2$, while the terms on the right side are of degrees not greater than d_0-1, therefore the terms on both sides are of degrees ranging over the interval $\delta=[d_p^-,d_0-1]$. Thus, we deduce that

$$-(\lambda^nV)_{+x}+[U,(\lambda^nV)_+]=\sum_{i\in\delta}f_i$$

for some $f_i\in\mathscr{G}_i\equiv\{x|\deg x=i,\, x\in\mathscr{G}\}$. Therefore, when we take $e_1(\lambda),\ldots,e_p(\lambda)$ as a basis of $\oplus_{i\in\delta}\mathscr{G}_i$, we could in general derive a hierarchy of integrable equations:

$$U_t-(\lambda^nV)_{+x}+[U,(\lambda^nV)_+]=0.$$

In this case, we have

$$p=(N^2-1)(d_0-d_p^-)=(N^2-1)(d_0-\tfrac{1}{2}d_p+\tfrac{1}{2}|d_p|). \tag{10}$$

It is easy to see that the number of potential functions increases quadratically with N. It is of practical importance to find reduced spectral problems where the number of potential functions is considerably less than (10). To this end, we need to consider various subalgebras of $\mathscr{G}$. It is known that the following relations hold:

$$[E_{ij}, E_{kl}]=0 \qquad (j\neq k, l\neq i)$$
$$[E_{ij}, E_{ji}]=E_{ii}-E_{jj}$$
$$[E_{ij}, E_{ki}]=-E_{kj} \qquad (k\neq j).$$

Set $H_i=E_{ii}-E_{i+1,i+1}$ and assume that $\mathscr{G}_1$ is a linear span of $\{H_i\otimes\lambda^{an+b_i}, E_{ij}\otimes\lambda^{an+b_{ij}}|i\neq j, n\in\mathbb{Z}$, where a is an integer greater than 1 and b_{ij}, b_i are integers such that $0\leqslant b_{ij}, b_i<a\}$. We can easily verify that $\mathscr{G}_1$ is a subalgebra of $\mathscr{G}$, thus

$$\begin{aligned} &b_i=0 \qquad (i=1,\ldots,N)\\ &b_{ij}+b_{ji}=0 \qquad (\mathrm{mod}\ a)\\ &b_{ij}+b_{ki}-b_{kj}=0 \qquad (\mathrm{mod}\ a). \end{aligned} \tag{11}$$

It is also easy to see that (11) is solvable. In what follows, we show that when $U\in\mathscr{G}_1$ and $\mathscr{G}_1$ is a subalgebra of $\mathscr{G}$, (3) has a series solution $V\in\mathscr{G}_1$. In fact, due to $U\in\mathscr{G}_1\subset\mathscr{G}$, we know from proposition 2 that there exists $V\in\mathscr{G}$ such that $V_x=[U, V]$. On the other hand, $\mathscr{G}$ can be decomposed as

$$\mathscr{G}=\mathscr{G}_1\oplus\mathscr{G}_2$$

where $\mathscr{G}_2$ is a linear span of

$$\bigcup_{\substack{1\leqslant k\leqslant a-1\\ 0\leqslant l\leqslant a-1\\ 1\leqslant i,j\leqslant N, n\in\mathbb{Z}}} \{H_i\otimes\lambda^{an+k}, (1-\delta_{l,b_{ij}})E_{ij}\otimes\lambda^{an+l}|i\neq j; k, l \text{ are integers}\}.$$

Thus, V can be rewritten as

$$V=V_1+V_2 \qquad V_i\in\mathscr{G}_i \qquad (i=1,2).$$

Note that

$$[\mathscr{G}_1,\mathscr{G}_1]\subset\mathscr{G}_1 \qquad [\mathscr{G}_1,\mathscr{G}_2]\subset\mathscr{G}_2$$

we have

$$V_x=[U, V] \Leftrightarrow \begin{cases} V_{1x}=[U, V_1]\\ V_{2x}=[U, V_2].\end{cases}$$

Thus we reach the following conclusion.

Proposition 3. Suppose $U=e_0(\lambda)+u_1e_1(\lambda)+\ldots+u_pe_p(\lambda)\in\mathscr{G}_1$ and $\mathscr{G}_1$ is a subalgebra of $\mathscr{G}$, i.e. (11) holds. Then there exists a series solution $V=\sum_{i=0}^{\infty}V_i\lambda^{-i}\in\mathscr{G}_1$ such that (3) holds and elements of matrix V_i are all pure polynomials of u_i $(i=1,\ldots,p)$ and their derivatives.

From proposition 3, we deduce that

$$-(\lambda^{an}V)_{+x}+[U, (\lambda^{an}V)_+]=(\lambda^{an}V)_{-x}-[U, (\lambda^{an}V)_-].$$

Similar to the above, we know that when we take $e_1(\lambda), \ldots, e_p(\lambda)$ as a basis of $\oplus_{i\in\delta}\mathscr{G}_{1i}$, we could in general derive a hierarchy of integrable equations, where $\delta=[d_p^-, d_0-1]$ and $\mathscr{G}_{1i}\equiv\{x|\deg x=i, x\in\mathscr{G}_1\}$. It is evident that

$$p<(N^2-1)(d_0-d_p^-)=(N^2-1)(d_0-\tfrac{1}{2}d_p+\tfrac{1}{2}|d_p|).$$

We will now give some examples. In order to fix the integral constant arising from calculation, we shall follow the homogeneous rank convention: *both sides of an equation have the same rank*, where the definition of rank is [7–10]

$$\begin{aligned}
&\operatorname{rank} x=\deg x && (x\in\mathscr{G})\\
&\operatorname{rank}(\lambda)=\deg(x\lambda)-\deg(x)\\
&\operatorname{rank}(u_i)=d_0-d_i && (i=1,\ldots,p)\\
&\operatorname{rank}(\partial)=d_0\\
&\operatorname{rank}(\beta)=0 && (\beta=\text{const}, \beta\neq 0).
\end{aligned}$$

Example 1. $N=3$. In this case, we have the following solution of (11):

$$a=3 \qquad b_{12}=b_{23}=b_{31}=1 \qquad b_{13}=b_{32}=b_{21}=2.$$

Furthermore, take $d_0=1$, $d_p=-2$ and

$$e_0(\lambda)=\begin{bmatrix}0&\lambda&0\\0&0&\lambda\\\lambda&0&0\end{bmatrix}.$$

Obviously, $p=8$. Thus we immediately get a new spectral problem,

$$\psi_x=\begin{bmatrix}u+v&\lambda+w_1\lambda^{-2}&r\lambda^{-1}\\\bar{p}\lambda^{-1}&-u&\lambda+w_2\lambda^{-2}\\\lambda+s\lambda^{-2}&q\lambda^{-1}&-v\end{bmatrix}\psi$$

which is considered in [11, 12].

Example 2. $N=2$. In this case, we have the following solution of (11):

$$a=2 \qquad b_{12}=b_{21}=1.$$

Furthermore, take $d_0=1$, $d_p=-1$ and

$$e_0(\lambda)=\begin{bmatrix}0&\lambda\\-\lambda&0\end{bmatrix} \qquad e_1(\lambda)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$e_2(\lambda)=\begin{bmatrix}0&1\\1&0\end{bmatrix}\lambda^{-1} \qquad e_3(\lambda)=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\lambda^{-1}.$$

We get the following spectral problem:

$$\psi_x=U\psi$$

where

$$U=\begin{bmatrix} u & \lambda-s\lambda^{-1}+v\lambda^{-1} \\ -\lambda+s\lambda^{-1}+v\lambda^{-1} & -u \end{bmatrix}.$$

Set

$$T=\begin{bmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{bmatrix}$$

we have

$$TUT^{-1}=\begin{bmatrix} -\mathrm{i}\lambda+\mathrm{i}\lambda^{-1}s & u+\mathrm{i}\lambda^{-1}v \\ u-\mathrm{i}\lambda^{-1}v & \mathrm{i}\lambda-\mathrm{i}\lambda^{-1}s \end{bmatrix}$$

which is simply the Boiti–Tu spectral problem [13].

Example 3. $N=3$. In this case, we have the following solution of (11):

$$a=4 \qquad b_{13}=b_{31}=2 \qquad b_{12}=b_{23}=1 \qquad b_{21}=b_{32}=3.$$

Furthermore, take $d_0=2$, $d_p=0$ and

$$e_0(\lambda)=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\lambda^2.$$

In this case, $p=4$. Thus, we get the following new spectral problem:

$$\psi_x=U\psi$$

where

$$U=\begin{bmatrix} r & \lambda u & \lambda^2 \\ 0 & s-r & \lambda q \\ \lambda^2 & 0 & -s \end{bmatrix}.$$

We take

$$V=\begin{bmatrix} a & b & c \\ d & e-a & f \\ g & h & -e \end{bmatrix}=\sum_{m\geqslant 0}\begin{bmatrix} a_m & b_m\lambda^{-3} & c_m\lambda^{-2} \\ d_m\lambda^{-1} & e_m-a_m & f_m\lambda^{-3} \\ g_m\lambda^{-2} & h_m\lambda^{-1} & -e_m \end{bmatrix}\lambda^{-4m}.$$

Then it follows from $V_x=[U, V]$ that

$$\begin{aligned} a_x&=\lambda ud+\lambda^2 g-\lambda^2 c \\ b_x&=(2r-s)b+\lambda u(e-2a)+\lambda^2 h \\ c_x&=(r+s)c+\lambda uf-\lambda^2 e-\lambda^2 a-\lambda qb \\ d_x&=(s-2r)d+\lambda qg-\lambda^2 f \\ e_x&=-\lambda^2 c+\lambda^2 g+\lambda qh \\ f_x&=(2s-r)f-\lambda^2 d+\lambda q(a-2e) \\ g_x&=\lambda^2 a-(s+r)g+\lambda^2 e \\ h_x&=\lambda^2 b+(r-2s)h-\lambda ug \end{aligned} \tag{12a}$$

or

$$
\begin{aligned}
a_{m_x} &= ud_m + g_m - c_m \\
b_{m_x} &= (2r-s)b_m + u(e_{m+1} - 2a_{m+1}) + h_{m+1} \\
c_{m_x} &= (r+s)c_m + uf_m - e_{m+1} - a_{m+1} - qb_m \\
d_{m_x} &= (s-2r)d_m + qg_m - f_m \\
e_{m_x} &= -c_m + g_m + qh_m \\
f_{m_x} &= (2s-r)f_m - d_{m+1} + q(a_{m+1} - 2e_{m+1}) \\
g_{m_x} &= a_{m+1} - (s+r)g_m + e_{m+1} \\
h_{m_x} &= b_m + (r-2s)h_m - ug_m .
\end{aligned}
\tag{12b}
$$

We now give the first few of a_m, b_m, c_m, d_m, e_m, f_m, g_m and h_m in two cases.

Case (a):

$$
\begin{aligned}
& a_0 = -e_0 = \beta_1 = \text{const} \neq 0 \qquad h_0 = 3\beta_1 u \qquad d_0 = 3\beta_1 q \qquad c_0 = 0 \\
& g_0 = -3\beta_1 uq \qquad b_0 = 3\beta_1 u_x + 3\beta_1(2s-r)u - 3\beta_1 u^2 q \\
& f_0 = -3\beta_1 q_x + 3\beta_1(s-2r)q - 3\beta_1 uq^2 \\
& a_1 = -3\beta_1 u_x q + 3\beta_1 u^2 q^2 + 3\beta_1 uq(r-2s) \\
& e_1 = -3\beta_1 uq_x - 3\beta_1 u^2 q^2 + 3\beta_1 uq(s-2r), \ldots .
\end{aligned}
$$

Case (b):

$$
\begin{aligned}
& a_0 = e_0 = h_0 = d_0 = 0 \qquad c_0 = g_0 = \beta_2 = \text{const} \neq 0 \\
& f_0 = \beta_2 q \qquad b_0 = \beta_2 u \qquad a_1 = \tfrac{1}{2}\beta_2(s+r) - \tfrac{1}{2}\beta_2 uq \\
& e_1 = \tfrac{1}{2}\beta_2 uq + \tfrac{1}{2}\beta_2(s+r) \qquad h_1 = \beta_2 u_x + \tfrac{3}{2}\beta_2 us - \tfrac{3}{2}\beta_2 ur - \tfrac{3}{2}\beta_2 u^2 q \\
& d_1 = -\beta_2 q_x + \tfrac{3}{2}\beta_2 qs - \tfrac{3}{2}\beta_2 qr - \tfrac{3}{2}\beta_2 uq^2, \ldots .
\end{aligned}
$$

In general, we can obtain recursively from (12b) all the a_m, b_m, c_m, d_m, e_m, f_m, g_m and h_m.

On the other hand, we have

$$
\begin{aligned}
-(\lambda^{4n}V)_{+x} + [U, (\lambda^{4n}V)_+]
&= -a_{n_x}H_1 - e_{n_x}H_2 - b_{n-1_x}\lambda E_{12} - f_{n-1_x}\lambda E_{23} - 2ua_n\lambda E_{12} \\
&\quad + ue_n\lambda E_{12} + q(a_n - 2e_n)\lambda E_{23} + 2rb_{n-1}\lambda E_{12} \\
&\quad - sb_{n-1}\lambda E_{12} - rf_{n-1}\lambda E_{23} + 2sf_{n-1}\lambda E_{23} .
\end{aligned}
$$

Therefore, we can deduce a hierarchy of equations:

$$
\begin{aligned}
u_t &= b_{n-1_x} + 2ua_n - ue_n - 2rb_{n-1} + sb_{n-1} = h_n \\
q_t &= f_{n-1_x} - qa_n + 2qe_n + rf_{n-1} - 2sf_{n-1} = -d_n \\
r_t &= a_{n_x} \\
s_t &= e_{n_x} .
\end{aligned}
\tag{13}
$$

We have two hierarchies of equations corresponding to two different choices of a_0, b_0, c_0, d_0, e_0, f_0, g_0 and h_0. In particular, for case (a), taking $n=1$, $\beta_1=\frac{1}{3}$ in (13), we have

$$u_t=u_{xx}+3(s-r)u_x+u(2s-r)_x-4uu_xq+(s-2r)(2s-r)u+6u^2q(r-s)+3u^3q^2$$

$$q_t=-q_{xx}+3(s-r)q_x+q(s-2r)_x-4uq_xq+(r-2s)(s-2r)q+6uq^2(s-r)-3u^2q^3$$

$$r_t=[-u_xq+u^2q^2+uq(r-2s)]_x$$

$$s_t=[-uq_x-u^2q^2+uq(s-2r)]_x.$$

For case (b), taking $\beta_2=1$, $n=1$ in (13), we have

$$u_t=u_x+\tfrac{3}{2}us-\tfrac{3}{2}ur-\tfrac{3}{2}u^2q$$

$$q_t=q_x-\tfrac{3}{2}qs+\tfrac{3}{2}qr+\tfrac{3}{2}uq^2$$

$$r_t=\tfrac{1}{2}(s+r)_x-\tfrac{1}{2}(uq)_x$$

$$s_t=\tfrac{1}{2}(s+r)_x+\tfrac{1}{2}(uq)_x.$$

Example 4. For general N, we have the following solution of (11):

$$a=N,\ b_{i,i+1}=1,\ b_{i,i+2}=2,\ \ldots,\ b_{i,i+N-1}=N-1$$

$$b_{i+1,i}=N-1,\ b_{i+2,i}=N-2,\ \ldots,\ b_{i+N-1,i}=1.$$

Furthermore, we take

$$e_0(\lambda)=\begin{bmatrix} 0 & 1 & & & 0 \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}\lambda$$

and choose $e_1(\lambda), \ldots, e_p(\lambda)$ as a basis of

$$\bigoplus_{l\in[-(N-1),0]} \mathscr{G}_{1l}.$$

In this case, $d_0=1$, $d_p=-(N-1)$. Thus, we can obtain an $N\times N$ spectral problem.

Example 5. $N=3$. In this case, we have the following solution of (11):

$$a=6 \qquad b_{13}=b_{31}=3 \qquad b_{12}=1 \qquad b_{23}=2 \qquad b_{21}=5 \qquad b_{32}=4.$$

Furthermore, we take $d_0=3$, $d_p=-1$ and

$$e_0(\lambda)=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\lambda^3.$$

In this case, $p=5$. Thus, we get the following spectral problem:

$$\psi_x = U\psi$$

where

$$U=\begin{bmatrix} u_3 & u_2\lambda & \lambda^3 \\ u_5\lambda^{-1} & u_4-u_3 & u_1\lambda^2 \\ \lambda^3 & 0 & -\lambda_4 \end{bmatrix}.$$

3. Further examples

In this section, we shall consider some other subalgebras of $\mathscr{G}=sl(N, C)\otimes C[\lambda, \lambda^{-1}]$ and the corresponding integrable systems. From [14], we know $sl(N, C)$ has the following subalgebras:

$$su(p, q):=\left\{\begin{pmatrix} Z_1 & Z_2 \\ \bar{Z}_2^T & Z_3 \end{pmatrix} \middle| \begin{matrix} Z_1, Z_3 \text{ skew Hermitian of order } p \text{ and } q \text{, respectively} \\ \operatorname{Tr} Z_1+\operatorname{Tr} Z_3=0, Z_2 \text{ arbitrary} \end{matrix}\right\}$$

$$su^*(2n):=\left\{\begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \middle| \begin{matrix} Z_1, Z_2\ n\times n \text{ complex matrices} \\ \operatorname{Tr} Z_1+\operatorname{Tr} \bar{Z}_1=0 \end{matrix}\right\}$$

$$sp(n, C):=\left\{\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^T \end{pmatrix} \middle| \begin{matrix} Z_i\ n\times n \text{ complex matrices} \\ Z_2 \text{ and } Z_3 \text{ symmetric} \end{matrix}\right\}$$

$$so^*(2n):=\left\{\begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \middle| \begin{matrix} Z_1, Z_2\ n\times n \text{ complex matrices} \\ Z_1 \text{ skew symmetric, } Z_2 \text{ Hermitian} \end{matrix}\right\}$$

$$sp(p, q):=\left\{\begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ \bar{Z}_{12}^T & Z_{22} & Z_{14}^T & Z_{24} \\ -\bar{Z}_{13} & \bar{Z}_{14} & \bar{Z}_{11} & -\bar{Z}_{12} \\ \bar{Z}_{14}^T & -\bar{Z}_{24} & -Z_{12}^T & \bar{Z}_{22} \end{pmatrix} \middle| \begin{matrix} Z_{ij} \text{ complex matrix; } Z_{11} \text{ and } Z_{13} \\ \text{of order } p,\ Z_{12} \text{ and } Z_{14}\ p\times q \text{ matrices,} \\ Z_{11} \text{ and } Z_{22} \text{ are skew Hermitian,} \\ Z_{13} \text{ and } Z_{24} \text{ are symmetric} \end{matrix}\right\}$$

and so on.

For each of $su(p, q)$, $su^*(2n)$, $sp(n, C)$, we could prove that there exists a corresponding $M\subset sl(N, C)$ such that

$$\begin{aligned} & sl(N, C)=K\oplus M \\ & [K, K]\subset K \qquad [K, M]\subset M \end{aligned} \tag{14}$$

where K is any one of $su(p, q)$, $su^*(2n)$ and $sp(n, C)$.

In fact, taking $su(p, q)$ as an example. Set

$$M:=\left\{\begin{pmatrix} Y_1 & Y_2 \\ -\bar{Y}_2^T & Y_3 \end{pmatrix} \middle| \begin{matrix} Y_1, Y_3 \text{ Hermitian of order } p \text{ and } q \text{, respectively} \\ p+q=N \qquad \operatorname{Tr} Y_1+\operatorname{Tr} Y_3=0 \end{matrix}\right\}.$$

It is easy to show that

$$\begin{aligned} & sl(N, C)=su(p, q)\oplus M \\ & [su(p, q), su(p, q)]\subset su(p, q) \\ & [su(p, q), M]\subset M. \end{aligned}$$

In addition, we have

$$[M, M] \subset su(p, q).$$

In what follows, we only consider two subalgebras of $sl(N, C)$: $su(p, q)$, $sp(p, q)$. For some other subalgebras, a similar discussion can be undertaken, but this will be provided elsewhere.

3.1. su(p, q)

First we consider $su(p, q)$. Obviously, $\mathscr{G}_2 = su(p, q) \otimes C[\lambda, \lambda^{-1}]$ is a subalgebra of $\mathscr{G} = sl(N, C) \otimes C[\lambda, \lambda^{-1}]$. From proposition 2 in section 2, we know that when the conditions of proposition 2 are fulfilled there exists a $V \in \mathscr{G}$ such that $V_x = [U, V]$. Thus, when $U \in su(p, q) \otimes C[\lambda, \lambda^{-1}] \subset \mathscr{G}$, we have $V \in \mathscr{G}$ such that $V_x = [U, V]$. Furthermore, we assume that the projection of V on $sl(N, C) \otimes C[\lambda, \lambda^{-1}]$ is non-trivial. It immediately follows from (10) that there exists a $V_1 \in su(p, q) \otimes C[\lambda, \lambda^{-1}]$ such that $V_{1_x} = [U, V_1]$.

In the following, we proceed to consider subalgebras of $\mathscr{G}_2$. Assume $q > 1$, then $su(p, q)$ can be rewritten as

$$su(p, \bar{q}, \bar{r}) := \left\{ \begin{pmatrix} Z_1 & Z_2 & Z_3 \\ \bar{Z}_2^T & Z_4 & Z_5 \\ \bar{Z}_3^T & -\bar{Z}_5^T & Z_6 \end{pmatrix} \middle| \begin{array}{l} Z_1, Z_4, Z_6 \text{ skew Hermitian of orders} \\ p, \bar{q} \text{ and } \bar{r}, \text{ respectively} \\ \mathrm{Tr}\, Z_1 + \mathrm{Tr}\, Z_4 + \mathrm{Tr}\, Z_6 = 0 \qquad \bar{q} + \bar{r} = q \\ Z_2, Z_3, Z_5 \text{ arbitrary} \end{array} \right\}.$$

Set

$$M_0 = \left\{ \begin{pmatrix} Z_1 & & \\ & Z_4 & \\ & & Z_6 \end{pmatrix} \middle| \begin{array}{l} Z_1, Z_4, Z_6 \text{ skew Hermitian of} \\ \text{orders } p, \bar{q} \text{ and } \bar{r}, \text{ respectively} \\ \mathrm{Tr}\, Z_1 + \mathrm{Tr}\, Z_4 + \mathrm{Tr}\, Z_6 = 0 \end{array} \right\}$$

$$M_1 = \left\{ \begin{pmatrix} 0 & Z_2 & 0 \\ \bar{Z}_2^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| Z_2 \text{ arbitrary} \right\}$$

$$M_2 = \left\{ \begin{pmatrix} 0 & 0 & Z_3 \\ 0 & 0 & 0 \\ \bar{Z}_3^T & 0 & 0 \end{pmatrix} \middle| Z_3 \text{ arbitrary} \right\}$$

$$M_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Z_5 \\ 0 & -\bar{Z}_5^T & 0 \end{pmatrix} \middle| Z_5 \text{ arbitrary} \right\}$$

and $\mathscr{G}_3$ is a linear span of $\{M_0 \otimes \lambda^{2n}, M_1 \otimes \lambda^{2n+b_1}, M_2 \otimes \lambda^{2n+b_2}, M_3 \otimes \lambda^{2n+b_3} | n \in Z, b_i$ either 0 or 1 $(i = 1, 2, 3)\}$. Obviously,

$$su(p, \bar{q}, \bar{r}) = M_0 \oplus M_1 \oplus M_2 \oplus M_3$$

and $\mathscr{G}_3$ is a subalgebra of $\mathscr{G}_2$, and thus

$$b_3 + b_1 - b_2 = 0 \qquad (\mathrm{mod}\ 2). \tag{15}$$

It is easy to give the following three solutions of (15):

Case I:

$$b_1=b_2=1 \qquad b_3=0$$

Case II:

$$b_2=b_3=1 \qquad b_1=0$$

Case III:

$$b_1=b_3=1 \qquad b_2=0.$$

Here we only consider case I and the simplest case: $p=\bar{q}=\bar{r}=1$. In this case, we take

$$e_0(\lambda)=\begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}.$$

Thus, $d_0=1$. Further, we take $d_p=0$ and obtain the following spectral problem:

$$\psi_x=U\psi=\begin{bmatrix} \mathrm{i}u & 0 & \lambda \\ 0 & \mathrm{i}v-\mathrm{i}u & w \\ \lambda & -w^* & -\mathrm{i}v \end{bmatrix}\psi$$

where u, v are real potential functions. * Denotes the conjugate. Obviously $U\in\mathscr{G}_2$. Set

$$V=\begin{bmatrix} \mathrm{i}a & d & e \\ d^* & -\mathrm{i}a+\mathrm{i}b & c \\ e^* & -c^* & -\mathrm{i}b \end{bmatrix}=\sum_{m\geqslant 0}\begin{bmatrix} \mathrm{i}a_m & d_m\lambda^{-1} & e_m\lambda^{-1} \\ d_m^*\lambda^{-1} & -\mathrm{i}a_m+\mathrm{i}b_m & c_m \\ e_m^*\lambda^{-1} & -c_m^* & -\mathrm{i}b_m \end{bmatrix}\lambda^{-2m}$$

where a, b are real. From $V_x=[U, V]$, we deduce that

$$\begin{aligned} &\mathrm{i}a_x=\lambda(e^*-e)\mathrm{i}b_x=\lambda(e-e^*)+wc^*-w^*c \\ &c_x=(-\mathrm{i}u+2\mathrm{i}v)c-\lambda d^*+w(\mathrm{i}a-2\mathrm{i}b) \\ &d_x=(2\mathrm{i}u-\mathrm{i}v)d-\lambda c^*+w^*e \\ &e_x=\mathrm{i}(u+v)e-\mathrm{i}\lambda(a+b)-dw \end{aligned} \tag{16a}$$

or

$$\begin{aligned} &\mathrm{i}a_{m_x}=e_m^*-e_m-\mathrm{i}b_{m_x}=e_m-e_m^*+wc_m^*-w^*c_m \\ &c_{m_x}=(-\mathrm{i}u+2\mathrm{i}v)c_m-d_m^*+w(\mathrm{i}a_m-2\mathrm{i}b_m) \\ &d_{m_x}=(2\mathrm{i}u-\mathrm{i}v)d_m-c_{m+1}^*+w^*e_m \\ &e_{m_x}=\mathrm{i}(u+v)e_m-\mathrm{i}(a_{m+1}+b_{m+1})-d_mw. \end{aligned} \tag{16b}$$

We now give the first few of a_m, b_m, c_m, d_m, e_m in two cases.

Case (a):

$$c_0=0 \qquad a_0=-b_0=\beta_1=\text{real const}\neq 0$$

$$d_0=-3\mathrm{i}\beta_1 w^* \qquad e_0=0 \qquad a_1=3\beta_1|w|^2 \qquad b_1=0$$

$$c_1=-3\mathrm{i}\beta_1 w_x+3\beta_1(2u-v)w, \ldots.$$

Case (b):

$$a_0=b_0=c_0=0 \qquad e_0=\beta_2=\text{real const}\neq 0$$

$$d_0=0 \qquad c_1=\beta_2 w \qquad a_1=\tfrac{1}{2}\beta_2(u+v) \qquad b_1=\tfrac{1}{2}\beta_2(u+v)$$

$$d_1=-\beta_2 w_x^*-\tfrac{3}{2}\mathrm{i}\beta_2 w^*(v-u)$$

$$e_1=-\frac{\mathrm{i}}{4}\beta_2(u+v)_x+\tfrac{1}{8}\beta_2(u+v)^2+\tfrac{1}{2}\beta_2|w|^2$$

$$\begin{aligned}c_2^*=\beta_2 w_{xx}^*&+\tfrac{3}{2}\mathrm{i}\beta_2 w_x^*(v-u)+\tfrac{3}{2}\mathrm{i}\beta_2 w^*(v-u)_x\\&-\mathrm{i}\beta_2(2u-v)w_x^*+\tfrac{3}{2}\beta_2 w^*(v-u)(2u-v)\\&+w^*[-\tfrac{1}{4}\mathrm{i}\beta_2(u+v)_x+\tfrac{1}{8}\beta_2(u+v)^2+\tfrac{1}{2}\beta_2|w|^2],\ \ldots .\end{aligned}$$

In general, we can obtain recursively from (16b) all the a_m, b_m, c_m, d_m, e_m. On the other hand, we have

$$\begin{aligned}-(\lambda^{2n}V)_{+x}&+[U,(\lambda^{2n}V)_+]\\&=-\begin{pmatrix}\mathrm{i}a_n & 0 & 0\\ 0 & -\mathrm{i}a_n+\mathrm{i}b_n & c_n\\ 0 & -c_n^* & -\mathrm{i}b_n\end{pmatrix}_x\\&\quad+\left[\begin{pmatrix}\mathrm{i}u & 0 & 0\\ 0 & -\mathrm{i}u+\mathrm{i}v & w\\ 0 & -w^* & -\mathrm{i}v\end{pmatrix},\begin{pmatrix}\mathrm{i}a_n & 0 & 0\\ 0 & -\mathrm{i}a_n+\mathrm{i}b_n & c_n\\ 0 & -c_n^* & -\mathrm{i}b_n\end{pmatrix}\right].\end{aligned}$$

Therefore, we can deduce a hierarchy of equations:

$$\begin{aligned}u_t&=a_{n_x}\\ v_t&=a_{n_x}\\ w_t&=c_{n_x}+c_n(\mathrm{i}u-2\mathrm{i}v)+2\mathrm{i}b_n w-\mathrm{i}a_n w=-d_n^* .\end{aligned}\tag{17a}$$

Thus, we can take $u=v$. In this case, (17) becomes

$$\begin{aligned}u_t&=a_{n_x}\\ w_t&=c_{n_x}-\mathrm{i}uc_n+2\mathrm{i}b_n w-\mathrm{i}a_n w=-d_n^* .\end{aligned}\tag{17b}$$

We have two hierarchies of equations corresponding to two different choices of a_0, b_0, c_0, d_0, e_0, f_0, g_0 and h_0. In particular, for case (a), taking $n=1$, $\beta_1=-\frac{1}{3}$ in (17b), we have

$$\begin{aligned}u_t&=-(|w|^2)_x\\ w_t&=\mathrm{i}w_{xx}-u_x w+\mathrm{i}u^2 w+\mathrm{i}w^2 w^*\end{aligned}$$

which is simply the Newell equation [15, 16]. For case (b), taking $\beta_2=1$, $n=2$ in (17b), we have

$$u_t=[\tfrac{1}{4}u_{xx}+\tfrac{3}{4}\mathrm{i}(w^*w_x-w_x^*w)+\tfrac{1}{2}u^3]_x$$

$$\begin{aligned}w_t=w_{xxx}&+\tfrac{3}{2}\mathrm{i}u_x w_x+2\mathrm{i}uw_{xx}+\tfrac{3}{4}\mathrm{i}u_{xx}w+\tfrac{1}{2}uu_x w\\&-\tfrac{1}{2}u^2 w_x-\tfrac{3}{4}w^2 w_x^*+\tfrac{9}{4}ww^*w_x+\mathrm{i}wu^3+\tfrac{5}{2}\mathrm{i}uw^2w^* .\end{aligned}$$

3.2. sp(p, q)

Set

$$M_1=\left\{\begin{pmatrix} Z_{11} & 0 & 0 & 0\\ 0 & Z_{22} & 0 & 0\\ 0 & 0 & \bar{Z}_{11} & 0\\ 0 & 0 & 0 & \bar{Z}_{22}\end{pmatrix}\middle| Z_{11}, Z_{22} \text{ skew Hermitian}\right\}$$

$$M_2=\left\{\begin{pmatrix} 0 & Z_{12} & 0 & 0\\ \bar{Z}_{12}^T & 0 & 0 & 0\\ 0 & 0 & 0 & -\bar{Z}_{12}\\ 0 & 0 & -Z_{12}^T & 0\end{pmatrix}\middle| Z_{14} \text{ of order } p\times q\right\}$$

$$M_3=\left\{\begin{pmatrix} 0 & 0 & 0 & Z_{14}\\ 0 & 0 & Z_{14}^T & 0\\ 0 & \bar{Z}_{14} & 0 & 0\\ \bar{Z}_{14}^T & 0 & 0 & 0\end{pmatrix}\middle| Z_{14} \text{ of order } p\times q\right\}$$

$$M_4=\left\{\begin{pmatrix} 0 & 0 & Z_{13} & 0\\ 0 & 0 & 0 & Z_{24}\\ -\bar{Z}_{13} & 0 & 0 & 0\\ 0 & -\bar{Z}_{24} & 0 & 0\end{pmatrix}\middle| Z_{13}, Z_{24} \text{ symmetric}\right\}.$$

Obviously,

$$sp(p,q)=M_1\oplus M_2\oplus M_3\oplus M_4.$$

Set $\mathscr{G}_4$ is a linear span of $\{M_1\otimes\lambda^{2n}, M_2\otimes\lambda^{2n+b_2}, M_3\otimes\lambda^{2n+b_3}, M_4\otimes\lambda^{2n+b_4} | n\in\mathbb{Z}, b_i$ is either 0 or 1 $(i=2,3,4)\}$. It is easy to show that $\mathscr{G}_4$ is a subalgebra of $sp(p,q)\otimes C[\lambda,\lambda^{-1}]$, thus

$$b_2+b_3-b_4=0 \qquad (\text{mod } 2). \tag{18}$$

We have the following three solutions of (18):

Case I:

$$b_3=b_4=1 \qquad b_2=0$$

Case II:

$$b_2=b_3=1 \qquad b_4=0$$

Case III:

$$b_2=b_4=1 \qquad b_3=0.$$

Here we only consider case I and the simplest case: $p=q=1$. In this case, we take

$$e_0(\lambda)=\begin{bmatrix} & & 1 & 0\\ & & 0 & 2\\ -1 & 0 & & \\ 0 & -2 & & \end{bmatrix}\lambda.$$

Thus, $d_0=1$. Further, we take $d_p=0$ and obtain the following spectral problem:

$$\psi_x=U\psi=\begin{bmatrix} \mathrm{i}u & w & \lambda & 0 \\ w^* & \mathrm{i}v & 0 & 2\lambda \\ -\lambda & 0 & -\mathrm{i}u & -w^* \\ 0 & -2\lambda & -w & -\mathrm{i}v \end{bmatrix}\psi$$

where u, v are real potential functions. Set

$$V=\begin{bmatrix} \mathrm{i}a & c & d & e \\ c^* & \mathrm{i}b & e & f \\ -d^* & e^* & -\mathrm{i}a & -c^* \\ e^* & -f^* & -c & -\mathrm{i}b \end{bmatrix}=\sum_{m\geq 0}\begin{bmatrix} \mathrm{i}a_m & c_m & d_m\lambda^{-1} & e_m\lambda^{-1} \\ c_m^* & \mathrm{i}b_m & e_m\lambda^{-1} & f_m\lambda^{-1} \\ -d_m\lambda^{-1} & e_m^*\lambda^{-1} & -\mathrm{i}a_m & -c_m^* \\ e_m^*\lambda^{-1} & -f_m^*\lambda^{-1} & -c_m & -\mathrm{i}b_m \end{bmatrix}\lambda^{-2m}$$

where a, b are real. From $V_x=[U, V]$, we deduce that

$$\begin{aligned}
\mathrm{i}a_x&=wc^*-w^*c+\lambda(d-d^*)\\
\mathrm{i}b_x&=w^*c-wc^*+2\lambda(f-f^*)\\
c_x&=(\mathrm{i}u-\mathrm{i}v)c+\mathrm{i}w(b-a)+\lambda e^*+2\lambda e\\
d_x&=2\mathrm{i}ud-2\mathrm{i}\lambda a+2we\\
e_x&=\mathrm{i}(u+v)e+wf-\lambda c^*-2\lambda c+w^*d\\
f_x&=2w^*e+2\mathrm{i}vf-4\mathrm{i}\lambda b
\end{aligned}\tag{19a}$$

or

$$\begin{aligned}
\mathrm{i}a_{m_x}&=wc_m^*-w^*c_m+(d_m-d_m^*)\\
\mathrm{i}b_{m_x}&=w^*c_m-wc_m^*+2(f_m-f_m^*)\\
c_{m_x}&=(\mathrm{i}u-\mathrm{i}v)c_m+\mathrm{i}w(b_m-a_m)+e_m^*+2e_m\\
d_{m_x}&=2\mathrm{i}ud_m-2\mathrm{i}a_{m+1}+2we_m\\
e_{m_x}&=\mathrm{i}(u+v)e_m+wf_m-c_{m+1}^*-2c_{m+1}+w^*d_m\\
f_{m_x}&=2w^*e_m+2\mathrm{i}vf_m-4\mathrm{i}b_{m+1}.
\end{aligned}\tag{19b}$$

Now we begin the recurrence process with the initial conditions

$$a_0=b_0=c_0=e_0=0 \qquad d_0=\alpha=\text{real const} \qquad f_0=\beta=\text{real const}$$
$$a_1=\alpha u \qquad b_1=\tfrac{1}{2}\beta v \qquad c_1=\tfrac{1}{3}(2\beta-\alpha)w+\tfrac{1}{3}(2\alpha-\beta)w^*, \ldots .$$

In general, we can obtain recursively from (19*b*) all the a_m, b_m, c_m, d_m, e_m, f_m. On the other hand, we have

$$\begin{aligned}
&-(\lambda^{2n}V)_{+x}+[U,(\lambda^{2n}V)_+]\\
&\quad=-\begin{pmatrix} \mathrm{i}a_n & c_n & 0 & 0 \\ c_n^* & \mathrm{i}b_n & 0 & 0 \\ 0 & 0 & -\mathrm{i}a_n & -c_n^* \\ 0 & 0 & -c_n & -\mathrm{i}b_n \end{pmatrix}_x\\
&\qquad+\left[\begin{pmatrix} \mathrm{i}u & w & 0 & 0 \\ w^* & \mathrm{i}v & 0 & 0 \\ 0 & 0 & -\mathrm{i}u & -w^* \\ 0 & 0 & -w & -\mathrm{i}v \end{pmatrix},\begin{pmatrix} \mathrm{i}a_n & c_n & 0 & 0 \\ c_n^* & \mathrm{i}b_n & 0 & 0 \\ 0 & 0 & -\mathrm{i}a_n & -c_n^* \\ 0 & 0 & -c_n & -\mathrm{i}b_n \end{pmatrix}\right].
\end{aligned}$$

Therefore, we can deduce a hierarchy of equations:

$$\begin{aligned} u_t &= a_{n_x} + \mathrm{i}wc_n^* - \mathrm{i}w^*c_n \\ v_t &= b_{n_x} - \mathrm{i}w^*c_n - \mathrm{i}wc_n^* \\ w_t &= c_{n_x} + (\mathrm{i}v - \mathrm{i}u)c_n - \mathrm{i}wb_n + \mathrm{i}wa_n. \end{aligned} \tag{20}$$

In particular, taking $n=1$ in (20), we have

$$\begin{aligned} u_t &= \alpha u_x - \tfrac{1}{3}\mathrm{i}(2\alpha - \beta)(w^{*2} - w^2) \\ v_t &= \tfrac{1}{2}\beta v_x + \tfrac{1}{3}\mathrm{i}(2\alpha - \beta)(w^{*2} - w^2) \end{aligned}$$

$$w_t = \tfrac{1}{3}(2\beta - \alpha)w_x + \tfrac{1}{3}(2\alpha - \beta)w_x^* - \mathrm{i}w(\tfrac{1}{2}\beta v - \alpha u) - \mathrm{i}(u - v)[\tfrac{1}{3}(2\beta - \alpha)w + \tfrac{1}{3}(2\alpha - \beta)w^*].$$

4. Concluding remarks

We have described an algorithm to generate integrable systems. As a result, many new integrable equations are derived in a systematic way. In recent years many papers have been dedicated to the subject and different methods have been constructed. It is worth noting that similar algorithms were essentially implemented in [17, 18].

In this paper, we only focus on deriving integrable systems. Naturally, the algebraic and geometric properties of these new equations could also be considered. For example, by using the trace identity proposed by Tu, we could easily establish the Hamiltonian structures of the new integrable equations derived in sections 2 and 3. Also the corresponding recursion operators of these integrable systems could be derived. Finally, it is possible that the new evolution equations derived in this paper will find physical applications.

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